Journal of Sound and Vibration (1997) **200**(4), 543–550

RESPONSE ERRORS OF NON-PROPORTIONALLY LIGHTLY DAMPED STRUCTURES

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(*Received* 28 *December* 1995, *and in final form* 20 *May* 1996)

1. INTRODUCTION

Determination of the damping properties of structures is an important problem, often investigated in the search for more satisfactory or accurate solutions. One of the most popular approaches is based on the addition of modal damping to the model. This method is used not only for the sake of analytical simplicity, but also because it is the most favorable way to measure or estimate. This is the way, for example, to estimate the material damping in finite element analysis of large flexible structures. The resulting damping matrix in modal co-ordinates is a diagonal one. The structural damping, for which the modal damping matrix is diagonal, is called classical, or proportional damping. In another approach a damping matrix proportional either to the mass, or to the stiffness matrix, or to both, is introduced. This technique produces proportional damping as well.

Some researchers have sought to replace the non-proportional damping with proportional damping. For example, Cronin [1] used a perturbation technique to approximate a solution for a non-proportionally damped system with harmonic excitation. Chung and Lee [2] applied a perturbation of the proportional damping system by the non-proportional damping matrix to determine the eigensolutions of weakly, non-proportionally damped systems. Bellos and Inman [3] proposed a simple decoupling technique that is not limited either by magnitude of damping or separation of the natural frequencies. The bounds on the response of a non-classically damped system under harmonic and transient excitation were investigated by Yae and Inman [4] and by Nicholson [5], respectively. Falszeghy [6] proposed a new orthogonal co-ordinate transformation that transforms a non-classically damped system into a form in which the error generated by discarding the off-diagonal terms of the damping matrix is minimized. However, the simplest and the most common approach to the problem is to replace the full damping matrix with a diagonal one by neglecting the off-diagonal terms of the non-proportional damping matrix. Several researchers have studied the error bounds generated by this simplified approach (see, for example, Shahruz and Ma [7], Shahruz, [8], Uwadia and Esfandiari [9], Hwang and Ma [10] and Bhaskar [11]). The bounds derived for the case of arbitrary damping are often too conservative for many practical applications (e.g., in example 2 of Shahruz [8], the actual maximal error was 4%, while the bounds spanned the range of $\pm 18\%$). This is the price paid for considering a general case, in which arbitrary values of damping are allowed. However, there is a considerably

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large number of cases in which the damping is rather small, and for these cases the damping evaluation simplifies significantly.

This letter is primarily concerned with the analysis of a particular but important case of lightly damped structures. For small damping the resonance peaks are distinctive, and the modal damping coefficients are much smaller than 1. The case of lightly and non-proportionally damped systems is quite often encountered by structural and mechanical engineers who deal with steel structures (Dimarogonas and Haddad [12]), rotating equipment (Ehrich [13]), space structures (Joshi [14] and Gawronski and Juang [15]), and large antennas and radar (Gawronski and Mellstrom [16] and Gawronski [17]). In this case, the less restrictive conditions can be obtained by applying the error analysis similarly to the Shahruz [8] approach. The observation that a linear system is practically uncoupled when the natural frequencies are adequately separated was reported by Hasselman [18], and discussed by Park *et al*. [19]. Here, the detailed error analysis, including the dependence of the error on the system parameters, is presented. Also, we show that the off-diagonal terms rarely cause a significant approximation error when damping is small. The condition of positive definiteness of the damping matrix sets the limits on its off-diagonal terms; they are bounded by the geometric average of the corresponding diagonal terms.

2. FLEXIBLE STRUCTURES WITH SMALL NON-PROPORTIONAL DAMPING

The equations of motion of an n-degree-of-freedom structure are given here in the modal co-ordinates

$$\ddot{q} + C\dot{q} + \Omega^2 q = Bf, \tag{1}$$

where q is the modal displacement vector, f is the forcing function, B is a column matrix describing the distribution of applied force in the modal representation, Ω is a diagonal matrix of natural frequencies, $\Omega = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_n)$ where ω_i is the ith natural frequency, and C is the matrix of modal damping, which is symmetric and positive definite.

The damping matrix C is a full rather than a diagonal matrix, and it can be decomposed into the diagonal (C_d) and off-diagonal (C_o) components

$$C = C_d + C_o. \tag{2}$$

Note that and $c_{dii} = c_{ii}$ and $c_{oij} = c_{ij}$, for $i \neq j$. A flexible structure is proportionally damped if $C_o = 0$, and it is non-proportionally (or non-classically) damped for $C_o \neq 0$. The displacement of the non-proportionally damped structure q is a solution of equation (1) which can be rewritten as

$$\ddot{q} + C_d\dot{q} + C_o\dot{q} + \Omega^2 q = Bf. \tag{3}$$

The displacement of the proportionally damped structure q_p is a solution of equation (1) for $C = C_d$, i.e.,

$$\ddot{q} + C_d\dot{q}_p + \Omega^2 q_p = Bf. \tag{4}$$

The matrix $Z = \operatorname{diag}(\xi_i)$ of modal damping factors ξ_i, $i = 1, 2, \ldots, n$, is obtained from the diagonal damping matrix as $Z = 0{\cdot}5\Omega^{-1}C_d$. The structure is considered to be lightly damped if the damping factors are small, i.e., if $\xi_i \ll 1$, for $i = 1, \ldots, n$. In further considerations we assume small damping, namely, that the damping matrix C_d satisfies the above condition.

Consider the natural frequency ω_i. We shall show that for a small damping ratio the kth and ith modal displacements at frequency ω_i are proportional to each other. Denote

$q_{pk}(\omega_i)$ as the kth modal displacement at natural frequency ω_i. Then $\kappa_{k,i}$ is a scalar representing the ratio of the ith and kth modal magnitudes at frequency ω_i, i.e.,

$$\kappa_{k,i} = |q_{pk}(\omega_i)|/|q_{pi}(\omega_i)|. \tag{5}$$

This scalar can be obtained in a closed form as a function of system parameters. Namely, from equation (4) one obtains

$$|q_{pi}(\omega)| = \frac{|b_i||f|}{\sqrt{(\omega_i^2-\omega^2)^2 + 4\xi_i^2\omega_i^2\omega^2}}, \tag{6}$$

where b_i is the ith row of B. Substituting equation (6) into equation (5) for $\omega = \omega_i$ and for $\omega = \omega_k$, with the dimensionless variables defined as $b_{k,i} = |b_k|/|b_i|$, and $\omega_{k,i} = \omega_k/\omega_i$, yields

$$\kappa_{k,i} = \frac{b_{k,i}}{\xi_{k,i}\sqrt{\omega_{k,i}^2 + (\omega_{k,i}^2 - 1)^2/4\xi_k^2}} \tag{7}$$

The plot of $\kappa_{k,i}$ versus $\omega_{k,i}$ and ξ_k for $\xi_{k,i} = b_{k,i} = 1$ is shown in Figure 1. The maximal value of $\kappa_{k,i}$ is 1 whenever the modal frequencies ω_k and ω_i are equal (i.e., for $\omega_{k,i} = 1$). From the plots it is clear that $\kappa_{k,i}$ (or modal coupling) remains small for the almost whole domain of $\omega_{k,i}$, except for the small neighborhood of $\omega_{k,i} \simeq 1$ (i.e., when modes are clustered). Also, for a given value of $\omega_{k,i}$, modal coupling is smaller if the damping is light.

3. ERROR ESTIMATES

Denote by e_i the ith modal error, $e_i = q_i - q_{pi}$. Subtracting equation (4) from equation (3), and introducing the modal factor ξ_i instead of the diagonal term of C_d (i.e., $c_{di} = 2\xi_i\omega_i$), one obtains

$$\ddot{e}_i + 2\xi_i\omega_i\dot{e}_i + \omega_i^2 e_i = -c_{oi}\dot{q}_i. \tag{8}$$

where c_{oi} is the ith row of C_o. A question arises when the error is small (compared to the system displacement q), that is, when the non-proportional part C_o can be ignored. Shahruz

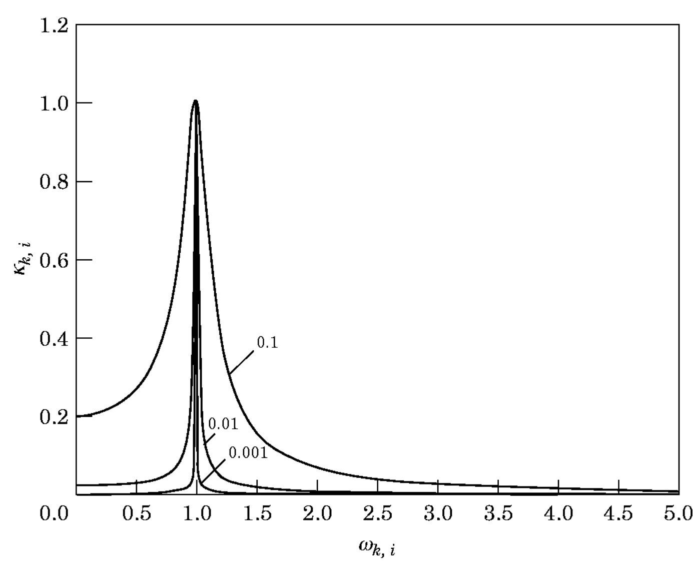

Figure 1. The ratio of the modal displacements with $\xi_{k,i} = b_{k,i} = 1$.

[7, 8] claims that it can be done if and only if the off-diagonal elements of the modal damping matrix are much smaller than the diagonal ones.

Let us define the factor s_i as

$$s_i = \kappa_i \sigma_i / 2\xi_i \omega_i, \tag{9}$$

where

$$\kappa_i = \max_{k \neq i}(\kappa_{i,k}),\ \sigma_i = \textstyle\sum_k |c_{oik}|$$

and c_{oik} is the kth term of the ith row, c_{oi}.

We will show that, for non-clustered natural frequencies and for the case of small damping, the ith mode error e_i is limited as follows

$$\frac{\|e_i(\omega)\|_2}{\|q_i(\omega)\|_2} = \frac{\|e_i(t)\|_2}{\|q_i(t)\|_2} < s_i \ll 1, \tag{10}$$

where the 2-norm of $q(\omega)$ is defined as

$$\|q\|_2^2 = (1/2\pi) \int_{-\infty}^{+\infty} |q(\omega)|^2 \, \mathrm{d}\omega.$$

First, we should show that $s_i \ll 1$. Indeed, note that the off-diagonal terms of the C matrix are at most of the order of the diagonal terms. This follows from the property of a positive-definite matrix C, which relates its off-diagonal and diagonal terms in the following manner: $c_{ij}^2 < c_{ii} c_{jj}$, for $i \neq j$ (see reference [20]). Thus, it follows from equation (9) that $\sigma_i / 2\xi_i \omega_i$ is of order 1. And since $\kappa_i \ll 1$, except for $\omega_{k,i} \simeq 1$ (i.e., when modes are clustered), it must be that $s_i \ll 1$.

Next, we prove that $\|e_i(\omega)\|_2 / \|q_i(\omega)\|_2 < s_i$. Rewriting equation (8) in the frequency domain,

$$e_i(\omega) = \frac{-\mathrm{j}\omega}{\omega_i^2 - \omega^2 + 2\mathrm{j}\xi_i \omega_i \omega} \sum_k c_{oik} q_{ik}(\omega). \tag{11}$$

Therefore,

$$|e_i(\omega_i)| = \frac{1}{2\xi_i \omega_i} \left| \sum_k c_{oik} q_{ik}(\omega_i) \right| \leqslant \frac{1}{2\xi_i \omega_i} \sum_k |c_{oik} q_{ik}(\omega_i)|. \tag{12a}$$

By further estimation one arrives at

$$|e_i(\omega_i)| \leqslant \frac{q_{max}}{2\xi_i \omega_i} \sum_{k \neq i} |c_{oik}| = \frac{q_{max} \sigma_i}{2\xi_i \omega_i}, \tag{12b}$$

where $q_{max} = \max_{k \neq i} |q_k(\omega_i)|$ is the largest modal displacement at frequency ω_i for $k \neq i$. However, from equation (5), the largest displacement is expressed in terms of the ith mode amplitude as

$$q_{max} = \kappa_i |q_{pi}(\omega_i)| \simeq \kappa_i |q_i(\omega_i)|, \tag{13}$$

where the use of the approximate equality $|q_{pi}(\omega_i)| \simeq |q_i(\omega_i)$ was justified for small s_i. Thus, substituting equation (13) into equation (12b) one obtains

$$\frac{|e_i(\omega_i)|}{|q_i(\omega_i)|} \leqslant \frac{\kappa_i \sigma_i}{2\xi_i \omega_i} = s_i. \tag{14}$$

Finally, note (see, for example, references [17] or [21]) that the 2-norms of e_i and q_i are proportional to their absolute values at the resonance frequency, i.e., $\| e_i \|_2 = \alpha_i |e_i(\omega_i)|$ and $\| q_i \|_2 = \alpha_i |q_i(\omega_i)|$, where $\alpha_i = \sqrt{0{\cdot}5\xi_i\omega_i}$. Due to this property one obtains equation (10). Finally, note that, from the Parseval theorem, the 2-norm also holds in the time domain: therefore the second condition in equation (10) is satisfied.

The above relationship implies that the ratio of the standard deviations of the error and of the response is limited by s_i, and in the case of the white noise input the error does not exceed the factor s_i. Since factor $\sigma_i/2\xi_i\omega_i$ is of order 1, the error is negligible if the factor κ_i is small, i.e., for almost all $\omega_{k,i}$ (see Figure 1). The large values of κ_i are observed only for the very close natural frequencies. Therefore, for separate natural frequencies the property (10) demonstrates that the off-diagonal elements of the damping matrix can be neglected regardless of their values.

4. EXAMPLES

Two examples are presented to illustrate the insignificance of the non-proportional damping terms and the strength of arguments expressed in the previous sections.

Example 1. Consider a system given by Sharuz [9], which is augmented here with the full damping matrix. It is represented in a modal form by

$$\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} 0{\cdot}0434 & 0{\cdot}0160 & 0{\cdot}0428 \\ 0{\cdot}0160 & 0{\cdot}0242 & 0{\cdot}0386 \\ 0{\cdot}0428 & 0{\cdot}0386 & 0{\cdot}0733 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4{\cdot}41 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1{\cdot}2 \\ 2{\cdot}5 \end{Bmatrix} \mathbf{1}(t),$$

where $\mathbf{1}(t)$, $t \geqslant 0$, denotes the unit step function. The system has the natural frequencies $\omega_1 = 2$, $\omega_2 = 2{\cdot}1$, and $\omega_3 = 3$, and the damping ratios $\xi_1 = 0{\cdot}011$, $\xi_2 = 0{\cdot}006$ and $\xi_3 = 0{\cdot}012$. Its fully populated modal matrix has all terms of the same order.

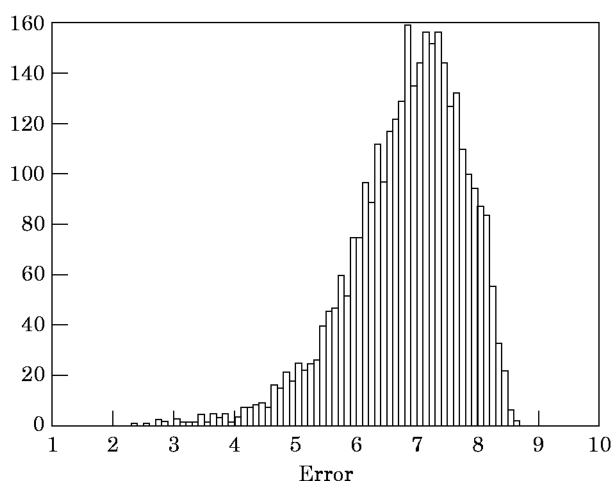

Figure 2. A histogram of the Euclidean error in the step response of the system in Example 1, for a randomly generated modal damping matrix.

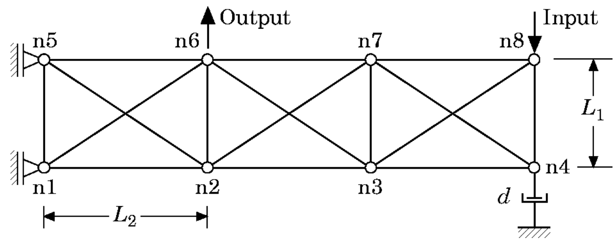

Figure 3. A truss structure with concentrated viscous damping.

A comparison between the exact solution and an approximated one, obtained by ignoring the non-proportional part of modal damping, has been calculated based on the Euclidean norm of the step response of the first component. For this example, the error was 4·36%. The distribution of the error for the 3500 samples of a randomly generated modal damping matrix is shown in Figure 2. The mean value and standard deviation for the displayed distribution are 6·8% and 0·93%, respectively. The considerably large error is observed due to the two closely spaced natural frequencies.

Example 2. Consider a flexible truss, presented in Figure 3. For this structure $L_1 = 70$ in and $L_2 = 100$ in each link has a cross-section area of 2 in^2, the material Young's modulus $E = 1 \times 10^6$ lb/in^2, and mass density $\rho = 2$ lbs^2/in^2. The system has 13 degrees of freedom. The structural damping is assumed to be 1% of critical damping and the concentrated viscous damping in a vertical direction at node 4 is ten times larger than the structural damping at this location.

The step force input is applied at node 8 and the response vibrations are measured at node 6 in the vertical direction. A comparison is made between the exact solution (i.e., with full damping matrix) and the approximate solution obtained by neglecting modal coupling in the damping matrix. According to Figure 4, the exact and approximate

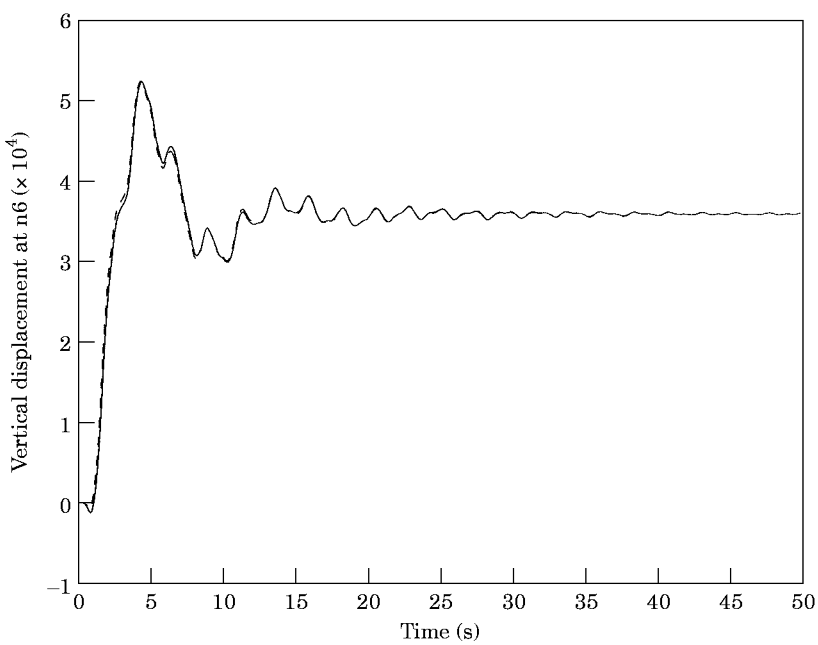

Figure 4. The exact and approximate displacement responses measured at node 6 for the truss structure: —, full damping; ––, proportional damping.

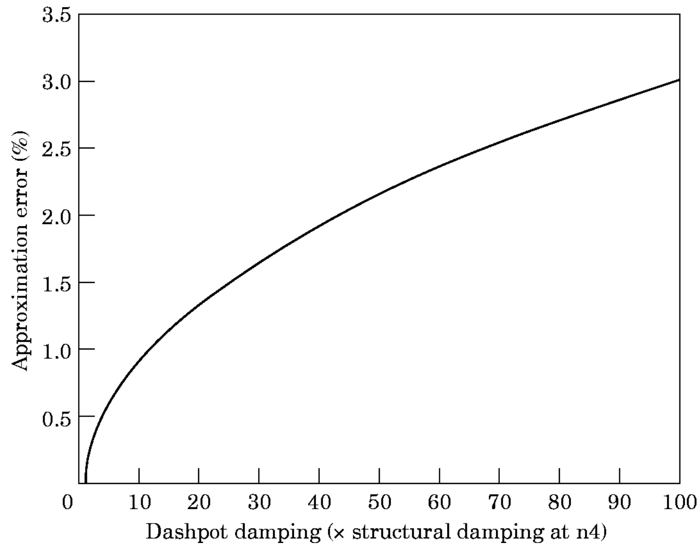

Figure 5. The approximation error versus the amount of concentrated viscous damping at node 4.

solutions are almost identical; the approximate solution has as much as 0·89% error compared to the exact one. Let C and C_d denote the full damping matrix and the diagonal part of the damping matrix, respectively; then the damping matrix diagonality index is defined as $di = \| C - C_d \| / \| C_d \| \times 100\%$ and is equal to 168% in this particular case. The plot of the approximation error versus viscous damping of the dashpot at node 4 is shown in Figure 5. Even for the viscous damping of the dashpot 100 times larger than the structural damping at the node 4, the approximation error does not exceed 3% based on the Euclidean norm.

5. CONCLUSIONS

In this letter we show that for flexible structures with small non-proportional damping, neglecting the off-diagonal terms of the modal damping matrix in most practical cases imposes negligible errors in the system dynamics. The Shahruz criterion, which allows neglecting the off-diagonal terms of the modal matrix, was relaxed. The requirement of the diagonal dominance of the damping matrix is not necessary in the case of small damping. If the natural frequencies are not clustered, one can ignore the off-diagonal terms of the damping matrix regardless of their value and introduce negligible error.

REFERENCES

1. D. L. CRONIN 1976 *Transactions of the American Society of Mechanical Engineers*, *Journal of Engineering for Industry* **98**, 43–47. Approximation for determining harmonically excited response of nonclassically damped systems.
2. K. R. CHUNG and C. W. LEE 1986 *Journal of Sound and Vibration* **111**, 37–50. Dynamic reanalysis of weakly non-proportionally damped systems.
3. J. BELLOS and D. J. INMAN 1990 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **112**, 194–201. Frequency response of nonproportionally damped, lumped parameter, linear dynamic systems.
4. K. H. YAE and D. J. INMAN 1987 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **54**, 419–423. Response bounds for linear underdamped systems.
5. D. W. NICHOLSON 1987 *Transactions of the American Society of Mechanical Engineers*, *Journal*

of Applied Mechanics **54**, 430–433. Response bounds for nonclassically damped mechanical systems under transient loads.

6. S. F. FELSZEGHY 1993 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **60**, 456–462. On uncoupling and solving the equations of motion of vibrating linear discrete systems.
7. S. M. SHAHRUZ and F. MA 1988 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **55**, 716–720. Approximate decoupling of the equations of motion of linear underdamped systems.
8. S. M. SHAHRUZ 1990 *Journal of Sound and Vibration* **136**, 51–64. Approximate decoupling of the equations of motion of damped linear systems.
9. F. E. UWADIA and R. S. ESFANDIARI 1990 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **57**, 423–433. Nonclassically damped dynamic systems: an iterative approach.
10. J. H. HWANG and F. MA 1993 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **60**, 695–701. On the approximate solution of nonclassically damped linear systems.
11. A. BHASKAR 1995 *Journal of Sound and Vibration* **184**, 59–72. Estimates of errors in the frequency response of non-classically damped systems.
12. A. D. DIMAROGONAS and S. HADDAD 1992 *Vibration for Engineers*. Englewood Cliffs, NJ: Prentice-Hall.
13. F. EHRICH 1992 *Handbook of Rotordynamics*. New York: McGraw-Hill.
14. S. M. JOSHI 1989 *Control of Large Flexible Space Structures*. Lecture Notes in Control and Information Sciences, 131. Berlin: Springer, Verlag.
15. W. GAWRONSKI and J. N. JUANG 1990 in *Control and Dynamic Systems* (C. T. Leondes, editor) **36**, 143–222. San Diego: Academic Press, Model reduction for flexible structures.
16. W. GAWRONSKI and J. A. MELLSTROM 1994 in *Control and Dynamic Systems*, (C. T. Leondes, editor) **63**, 289–412. San Diego: Academic Press. Control and dynamics of the Deep Space Network antennas.
17. W. GAWRONSKI 1996 *Balanced Control of Flexible Structures*. Lecture Notes in Control and Information Sciences, 211. London: Springer–Verlag.
18. T. K. HASSELMAN 1976, *American Institute of Aeronautics and Astronautics Journal* **14**, 1627–1628. Modal analysis in lightly damped structures.
19. I. W. PARK, J. S. KIM and F. MA, 1994, *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **61**, 77–83. Characteristics of modal coupling in nonclassically damped systems under harmonic excitation.
20. R. A. HORN and C. R. JOHNSON 1985 *Matrix Analysis*. Cambridge: Cambridge University Press.
21. J. S. BENDAT and A. G. PIERSOL 1993 *Engineering Applications of Correlation and Spectral Analysis*. New York: Wiley-Interscience.